THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2014–2015)
Introduction to Topology
Exercise 9 Compact Hausdorff

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $(X, \mathcal{T})$ be a Hausdorff space such that every proper subset of X is compact. Show that the topology is discrete. Do you think the converse is true?

2. Let $(X, \mathcal{T})$ be a compact Hausdorff space. Prove that for every $x \in X$ and every open set U containing x, there is a compact set K such that $x \in \mathrm{Int}(K) \subset K \subset U$ (i.e. compact neighborhoods form a local base). As a consequence, the conclusion is true also for locally compact Hausdorff space.

   Give an example of a compact non-Hausdorff space such that there is a point which has no local base formed by compact neighborhoods.

3. Let Y be compact Hausdorff. For a mapping $f\colon X \to Y$, define
$$G = \{\, (x, f(x)) \in X \times Y \;:\; x \in X \,\} .$$
   Prove that f is continuous if and only if G is a closed subset of $X \times Y$.

4. It is known that a continuous bijection $f\colon X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism. Give examples to show that the compactness of X and Hausdorff property of Y cannot be dropped.

5. Show that an infinite set X with cofinite topology is T_1 but not T_2. What if the set is finite?

6. Let $f, g\colon X \to Y$ be continuous mappings. What is the natural requirement for Y if you need one of the following:
   (a) The set $\{\, x \in X : f(x) = y_0 \,\}$ is closed for every $y_0 \in Y$.
   (b) The set $\{\, x \in X : f(x) = g(x) \,\}$ is closed.

7. A metric d is defined on $n \times n$ matrices by $d(A, B) = \left[\mathrm{tr}((A - B)(A - B)^T)\right]^{1/2}$. Convince yourself that the orthogonal matrices $\mathrm{O}(n)$ is compact but $\mathrm{SL}(n)$ is not, where
$$\begin{aligned} Q \in \mathrm{O}(n) \quad &\text{iff} \qquad QQ^T = \text{identity}, \\ A \in \mathrm{SL}(n) \quad &\text{iff} \qquad \det(A) = 1\,. \end{aligned}$$

8. Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the standard $\mathbb{S}^n$.

9. Let $(X, \mathcal{T})$ be a Hausdorff space. Show that the following two statements are equivalent.

   - Each $x \in X$ has a compact neighborhood N containing x.
   - For each $x \in X$ and each neighborhood U of x, there is a compact set $K \subset U \subset X$ such that $x \in \operatorname{Int}(K)$. That is, compact neighborhoods form a local base at each $x \in X$.